Math 347: Homework 3
Due on: Sep. 26, 2018

1. (*) Use induction to prove that a set of n elements has 2^n subsets.

2. (*) A function $f : \mathbb{R} \to \mathbb{R}$ is called a *derivation*[1] if it satisfies the following

$$f(xy) = xf(y) + yf(x), \quad \text{and} \quad f(x + y) = f(x) + f(y).$$

   (i) Prove that any derivation f satisfies

$$f(1) = 0 \quad \text{and} \quad f(u^n) = nu^{n-1}f(u)$$

   for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

   (ii) Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, one can define a new function

$$[f, g](x) = f(g(x)) - g(f(x)).$$

   Check that if f and g are derivations so is $[f, g]$.

   (iii) Suppose that $f_1, \dots, f_n$ are derivations from $\mathbb{R}$ to $\mathbb{R}$. Is $[f_1, [f_2, \cdots [f_{n-1}, f_n] \cdots]$ a derivation?

3. (*) Let $f(n) = n^2 - 8n + 18$. For which $n \in \mathbb{N}$ is $f(n) > f(n-1)$? (State and prove your result.)

4. Use induction on n to prove that a polynomial $f(x)$ of degree n with real coefficients has at most n roots[2].

5. (*) Prove that any integer number n is a prime[3] or the product of primes.

6. A subset S of $\mathbb{R}^2$ is *convex* if for every two points $x, y \in S$, the line segment joining x and y is contained in S.

   Prove that for any $n \geq 3$, the sum of the internal angles of a convex polygon with n vertices is $180 \cdot (n - 2)$ degrees.

7. Starting with 0, two players alternatively add $1, 2$ or 3 to a single running total. The player who first brings the total to at least 1000 wins. Prove that the second player has a strategy to win against any strategy for the first player.

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[1] This is not exactly the meaning of derivation you learned in Calculus, though it is related. The identity imposed on f is called the Leibniz rule.

[2] Recall that a *root of* $f(x)$ is any real number $a \in \mathbb{R}$, s.t. $f(a) = 0$.

[3] Recall that a prime number is one that is only divisible by 1 and itself.